CLASSICAL VERSUS QUANTUM SYMMETRIES FOR TODA THEORIES WITH A NONTRIVIAL BOUNDARY PERTURBATION

S. Penati *, A. Refolli † and D. Zanon ‡

Dipartimento di Fisica dell' Università di Milano and
INFN, Sezione di Milano, Via Celoria 16, I-20133 Milano, Italy

Abstract

In this paper we present a detailed study of the quantum conservation laws for Toda field theories defined on the half plane in the presence of a boundary perturbation. We show that total derivative terms added to the currents, while irrelevant at the classical level, become important at the quantum level and in general modify significantly the quantum boundary conservation. We consider the first nontrivial higher–spin currents for the simply laced $a_n^{(1)}$ Toda theories: we find that the spin–three current leads to a quantum conserved charge only if the boundary potential is appropriately redefined through a finite renormalization. Contrary to the expectation we demonstrate instead that at spin four the classical symmetry does not survive quantization and we suspect that this feature will persist at higher–spin levels. Finally we examine the first nontrivial conservations at spin four for the $d_3^{(2)}$ and $c_2^{(1)}$ nonsimply laced Toda theories. In these cases the addition of total derivative terms to the bulk currents is *necessary* but sufficient to ensure the existence of corresponding quantum exact conserved charges.

IFUM–522–FT December 1995

*E–mail address: penati@mi.infn.it
†E–mail address: refolli@mite35.mi.infn.it
‡E–mail address: zanon@mi.infn.it

1 Introduction

Certain classes of field theories defined on the two–dimensional plane possess an infinite number of conservation laws which generalize the energy–momentum conservation. These theories have been extensively studied primarily because the existence of such higher–spin conserved charges ensures that their exact S–matrix can be constructed [1, 2] and therefore all the on–shell properties can be determined. In general it has been found [3, 4] that whenever the system exhibits a classical symmetry, the corresponding conservation law can be implemented at the quantum level simply redefining appropriately the current via the addition of quantum corrections. In this way quantum integrability has been established for all affine Toda theories in their bosonic [4] as well as in their supersymmetric [5] version. The method which more efficiently allows to construct the higher–spin currents at the quantum level is massless perturbation theory. This approach, which is similar in spirit to perturbed conformal field theory, has the advantage to be applicable straightforwardly and to give exact results to all orders in perturbation theory.

Recently there has been much interest in studying these same theories defined not on the whole plane but on half of it, i.e. on a manifold with boundary [6, 7]. In this case one can consider the system in the presence of a nontrivial boundary perturbation and it turns out that many physical interesting phenomena can be described in this fashion [6, 8]. A natural question thus arises: how much of the integrability properties of the original model which locally are still valid in the interior region, do survive as global symmetries of the theory in the presence of the boundary? It is easy to show that in order to construct an integral of motion in terms of the currents conserved in the bulk, there must be no momentum flowing through the boundary or at most the momentum evaluated at the boundary has to reduce to a total time–derivative term. At the classical level this analysis has been carried out for all Toda theories and it has been shown [9, 10] that in general the higher–spin charges are conserved if the boundary perturbation is chosen appropriately.

These conservation laws if realized at the quantum level guarantee absence of particle production and the factorization property of the S matrix. The sine–Gordon model has been studied quite thoroughly [7, 11, 12]. One finds that: i) a renormalization of the currents is sufficient to reabsorb the quantum anomalies; ii) the most general boundary potential allowed by the requirement of quantum integrability contains two free parameters; iii) the exact S–matrix has been constructed and its connection with the underlying field theory description is understood.

For the other Toda theories the quantum analysis has not been completed yet [13].

Using a generalization of the massless perturbation approach which is standard for systems without boundary we have started a study of the quantum boundary conservations for models based on simply laced [11] and nonsimply laced [14] Lie algebras. In this paper we continue in this task. In the next section we set our notations and review the general procedure. In particular we derive the equations that need be satisfied in order to ensure the existence of a conserved charge at spin three and spin four level. In section 3 these results are applied to the specific examples of the $a_n^{(1)}$ Toda theories. We find that at spin three a quantum conserved charge exists if the classical boundary potential is suitably modified by a finite renormalization. Instead at spin four, somewhat unexpectedly, the classical symmetry does not survive quantization and there are indications that the charge conservation is broken by true anomalies at higher–spin levels too. In section 4 we study the first nontrivial currents (at spin four) for the $d_3^{(2)}$ and $c_2^{(1)}$ nonsimply laced Toda theories. Here we show that a quantum exact symmetry is realized only if total derivative terms are added to the bulk currents. No redefinition of the classical boundary potential is necessary in these cases. Finally in the last section we draw our conclusions and make some closing remarks.

2 Quantum charge conservation: the general procedure

We are interested in determining higher conservation laws for Toda–like systems defined in the upper–half plane. To this end it is convenient to work in euclidean space with complex coordinates

$$x = \frac{x_0 + ix_1}{\sqrt{2}} \qquad \bar{x} = \frac{x_0 - ix_1}{\sqrt{2}} \tag{2.1}$$

and corresponding derivatives

$$\partial \equiv \partial_x = \frac{1}{\sqrt{2}}(\partial_0 - i\partial_1) \qquad \bar{\partial} \equiv \partial_{\bar{x}} = \frac{1}{\sqrt{2}}(\partial_0 + i\partial_1) \qquad \Box = 2\partial\bar{\partial} \tag{2.2}$$

We restrict our attention to theories defined by the following action

$$\mathcal{S} = \frac{1}{\beta^2}\int d^2x \left\{\theta(x_1)\left[\frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial_\mu\vec{\phi} + V\right] - \delta(x_1)B\right\} \tag{2.3}$$

where B denotes the perturbation at the boundary and V is the affine Toda potential

$$V = \sum_{j=0}^{N} q_j e^{\vec{\alpha}_j\cdot\vec{\phi}} \tag{2.4}$$

The Toda theories under consideration are based on a Lie algebra $\mathcal{G}$ of rank N and we are using the standard notation for the simple roots α_j, $(j = 1, \cdots, N)$, $\alpha_0 = -\sum_{j=1}^{N} q_j \alpha_j$, with q_j the Kac labels $(q_0 = 1)$. From the action in (2.3) we can immediately derive the equations of motion in the bulk region

$$\Box \vec{\phi} = \sum_{j=0}^{N} q_j \vec{\alpha}_j e^{\vec{\alpha}_j \cdot \vec{\phi}} \tag{2.5}$$

and at the boundary

$$\left. \frac{\partial \phi_a}{\partial x_1} \right|_{x_1=0} = -\frac{\partial B}{\partial \phi_a} \tag{2.6}$$

It is well known [15] that for affine Toda theories one can construct an infinite set of classical currents and show that they are on–shell conserved in the interior region $x_1 > 0$. The classical conservation laws can be written as

$$\bar{\partial} J^{(n)} + \partial \Theta^{(n)} = 0 \qquad\qquad \partial \tilde{J}^{(n)} + \bar{\partial} \tilde{\Theta}^{(n)} = 0 \tag{2.7}$$

where $J^{(n)}$, $\Theta^{(n)}$ denote the two components of a spin $+n$ current, and $\tilde{J}^{(n)}$, $\tilde{\Theta}^{(n)}$ the corresponding ones for the spin $-n$ current. In terms of these currents one would like to define a conserved charge and this is feasible if the following conditions are met at the boundary

$$\left. J_1^{(n)} \right|_{x_1=0} \equiv i \left(J^{(n)} - \tilde{J}^{(n)} - \Theta^{(n)} + \tilde{\Theta}^{(n)} \right)\Big|_{x_1=0} = -\partial_0 \Sigma_0^{(n)} \tag{2.8}$$

with $\Sigma_0^{(n)}$ a local function of the fields evaluated at $x_1 = 0$. Indeed, if this is the case, an integral of motion is given by

$$q^{(n-1)} = \int_0^{+\infty} dx_1 J_0^{(n)} \quad + \quad \Sigma_0^{(n)} \tag{2.9}$$

where $J_0^{(n)} = J^{(n)} + \tilde{J}^{(n)} + \Theta^{(n)} + \tilde{\Theta}^{(n)}$. It has been shown [7, 9, 10] that the conditions in eq.(2.8) are satisfied if one restricts the class of boundary potentials to

$$B = \sum_{j=0}^{N} d_j e^{\frac{1}{2} \vec{\alpha}_j \cdot \vec{\phi}} \tag{2.10}$$

with appropriate coefficients d_j.

We note that equivalent sets of currents can be obtained through the addition to $J^{(n)}$, $\Theta^{(n)}$, $\tilde{J}^{(n)}$, $\tilde{\Theta}^{(n)}$ of total derivative terms

$$\begin{aligned} J^{(n)} &\to J^{(n)} + \partial U \qquad , \qquad & \Theta^{(n)} &\to \Theta^{(n)} - \bar{\partial} U \\ \tilde{J}^{(n)} &\to \tilde{J}^{(n)} + \bar{\partial} \tilde{U} \qquad , \qquad & \tilde{\Theta}^{(n)} &\to \tilde{\Theta}^{(n)} - \partial \tilde{U} \end{aligned} \tag{2.11}$$

Indeed the bulk conservation equations in (2.7) are not modified. Moreover it is immediate to verify that the charges constructed with the redefined quantities

$$
\begin{aligned}
&J_0^{(n)} \to J_0^{(n)} - i\sqrt{2}\,\partial_1(U-\tilde{U}) \qquad\qquad J_1^{(n)} \to J_1^{(n)} + i\sqrt{2}\,\partial_0(U-\tilde{U})\\
&\Sigma_0^{(n)} \to \Sigma_0^{(n)} - i\sqrt{2}\,(U-\tilde{U})\Big|_{x_1=0}
\end{aligned}
\tag{2.12}
$$

are still conserved and actually coincide with the ones in (2.9). The net result is that at the classical level there is the freedom of adding total derivatives to the currents, but these terms are irrelevant as far as the conservation laws are concerned. This will not be the case once quantum corrections will be included.

Thus, in order to proceed further we discuss now the issue of quantum conservation. As anticipated in the introduction the approach best suited to treat the problem exactly, to all–loop orders, is massless perturbation theory. This method treats the whole exponential in (2.4) as interaction terms without separating the quadratic parts which would correspond to mass terms. In this way calculations beyond one loop are much simpler as compared to the corresponding ones in a massive perturbative approach. We compute in x–space, using for the action in (2.3) massless propagators defined in the upper–half plane as

$$
G_{ij}(x,x') = -\frac{\beta^2}{4\pi}\delta_{ij}\left[\log 2|x-x'|^2 + \log 2|x-\bar{x}'|^2\right] \tag{2.13}
$$

and an interaction term $\mathcal{S}_i \equiv \mathcal{S}_i^V + \mathcal{S}_i^B$, where $\mathcal{S}_i^V = \frac{1}{\beta^2}\int_{-\infty}^{+\infty} dx_0 \int_0^{+\infty} dx_1 V$, with V the affine Toda potential (2.4) and $\mathcal{S}_i^B = -\frac{1}{\beta^2}\int_{-\infty}^{+\infty} dx_0 B$, with B the boundary perturbation (2.10).

At this point the classical conservation equations in (2.7) and (2.8) can be reexpressed in perturbation theory as

$$
\bar{\partial}\left\langle J^{(n)}(x,\bar{x})\right\rangle \equiv \bar{\partial}\left\langle J^{(n)}(x,\bar{x})\; e^{-\mathcal{S}_i^V}\right\rangle_0 = \; -\,\partial\left\langle \Theta^{(n)}\right\rangle \tag{2.14}
$$

for $x_1 > 0$ and

$$
\left\langle J_1^{(n)}(x,\bar{x})\right\rangle\Big|_{x_1=0} \equiv \left\langle J_1^{(n)}(x,\bar{x})\; e^{-\mathcal{S}_i}\right\rangle_0\Big|_{x_1=0} = \; -\,\partial_0\left\langle \Sigma_0^{(n)}\right\rangle \tag{2.15}
$$

at $x_1 = 0$. Classical results correspond to tree level calculations, while quantum corrections are given by loop contributions. Normal ordering of the exponentials in V and B is always understood so that no ultraviolet divergences are produced.

At spin n we consider currents of the form

$$
J^{(n)} = \sum c_{ab}\partial^{a_1}\phi_{b_1}\ldots\partial^{a_s}\phi_{b_s} \tag{2.16}
$$

where $a \equiv (a_1, \cdots, a_s)$ and $\sum a_i = n$. The coefficients c_{ab} are given in general by a power expansion in the coupling constant β^2

$$c_{ab} = c_{ab}^{(0)} + \beta^2 c_{ab}^{(1)} + \dots \tag{2.17}$$

The zeroth order term, $c_{ab}^{(0)}$, must be such that the classical conservation law in (2.7) is satisfied.

We evaluate quantum corrections in the interior region Wick contracting the $J^{(n)}$ current with the exponential in (2.14). Since the current contains terms with at most n $\partial\phi$ factors, it is clear that we need compute at most up to $n-1$ loops. Among all the contributions we want to select the local terms. The ones that are expressible as total ∂–derivatives contribute directly to the quantum trace in (2.14); the rest has to vanish if the current conservation is anomaly free. Therefore one must determine the yet unknown quantum coefficients in (2.17) in order to cancel these potentially anomalous terms. This procedure has been applied successfully in several examples [4, 5].

The actual calculation in (2.14) is simplified by the observation that since we are interested only in local contributions it is sufficient to expand the exponential to first order in S_i^V. Indeed Wick contractions of the current with the interaction produce in general a sum of terms of the form

$$\bar{\partial}_x \int d^2w \ \ \mathcal{M}(x,\bar{x}) \left[\frac{1}{(x-w)^k} + \frac{1}{(x-\bar{w})^k} \right] \mathcal{N}(w,\bar{w}) \tag{2.18}$$

where $\mathcal{M}$, $\mathcal{N}$ are products of the fields and their ∂–derivatives and the integration is performed in the upper plane. Local expressions are obtained using in the half plane the relation

$$\bar{\partial}_x \frac{1}{(x-w)^k} = \frac{2\pi}{(k-1)!} \partial_w^{k-1} \delta^{(2)}(x-w) \tag{2.19}$$

Since only one $\bar{\partial}$ is present, only one interaction factor (one integration) can appear if we want to obtain a local result. In this way we determine the contributions to the quantum trace and the renormalization of the classical current to all orders of perturbation theory. This part of the calculation is performed in the interior region and in a certain sense it is preliminary and preparatory for the actual check of the charge conservation in the presence of the boundary perturbation.

At the boundary $x_1 = 0$ we have to consider eq.(2.15). Using the quantum expressions just obtained in the bulk for $J^{(n)}$ and $\Theta^{(n)}$ we compute $J_1^{(n)} = i(J^{(n)} - \tilde{J}^{(n)} - \Theta^{(n)} + \tilde{\Theta}^{(n)})$ and then we evaluate its expectation value at $x_1 = 0$ as in (2.15). The aim is to isolate local terms which are not ∂_0–derivatives and see if they correspond to true anomalies. In this case the calculation is complicated by the fact that local contributions might

arise from higher–order terms in the expansion of the interaction potential, given now by the complete sum of V in the bulk and B at the boundary. Typically expanding the exponential in (2.15) to first order in $\mathcal{S}_i^B$ the following structures are produced

$$\lim_{x_1\to 0}\int_{-\infty}^{+\infty} dw_0 \left[\mathcal{P}(x,\bar{x})\left(\frac{1}{(x-w)^k}+\frac{1}{(x-\bar{w})^k}\right)-\tilde{\mathcal{P}}(x,\bar{x})\left(\frac{1}{(\bar{x}-\bar{w})^k}+\frac{1}{(\bar{x}-w)^k}\right)\right]\mathcal{Q}(w_0) \tag{2.20}$$

where $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are functions of $\partial^k\phi$ and $\bar{\partial}^k\phi$ respectively. Since $w=\bar{w}$, being $w_1=0$, the above expression can be written as

$$2(\sqrt{2})^k \lim_{x_1\to 0}\int_{-\infty}^{+\infty} dw_0 \left[\mathcal{P}(x,\bar{x})\frac{1}{(x_0-w_0+ix_1)^k}-\tilde{\mathcal{P}}(x,\bar{x})\frac{1}{(x_0-w_0-ix_1)^k}\right]\mathcal{Q}(w_0) \tag{2.21}$$

Local boundary contributions are obtained selecting in $\mathcal{P}$ and $\tilde{\mathcal{P}}$ terms which are equal and making use of the following relation

$$\lim_{x_1\to 0^+}\left(\frac{1}{(x_0-w_0-ix_1)^k}-\frac{1}{(x_0-w_0+ix_1)^k}\right)=\frac{2\pi i}{(k-1)!}\partial_{w_0}^{k-1}\delta^{(1)}(x_0-w_0) \tag{2.22}$$

Repeating the same procedure, it is clear that Wick contractions with higher–order factors in the expansion of the boundary interaction give rise to local contributions whenever the number of one–dimensional $\delta^{(1)}$–functions produced in the limit $x_1\to 0$ equals the number of integrations.

We also have to take into account terms from the expansion of the bulk potential and/or from mixed factors of the bulk and the boundary potentials. Such a computation requires in general a lengthy algebraic effort. We present an explicit example in Appendix A.

As mentioned above anomalous boundary contributions would correspond to local terms which cannot be written as ∂_0–derivatives of suitable expressions. Now we want to show that at the quantum level total derivative terms added to the current and to the trace might influence these potential anomalies.

The addition of a ∂U term to the $J^{(n)}$ current modifies the quantum conservation condition in the bulk by a term $\bar{\partial}\langle\partial U\rangle=\partial\bar{\partial}\langle U\rangle$. Obviously, being the result automatically in the form of a total ∂–derivative, no anomaly is produced in the interior region and the local terms obtained from $\bar{\partial}\langle U\rangle$ will all contribute to the quantum trace. Now, while the tree level (classical) contributions are equal to $\bar{\partial}U$, the loop (quantum) corrections are not expressible in general as $\bar{\partial}$–derivatives. Consequently these terms might lead to quantum corrections in $J_1^{(n)}$ which are not ∂_0–derivatives and therefore affect the boundary condition (2.15) in a nontrivial manner. Since these corrections will play a relevant role in the following sections, we illustrate this point in detail with an example.

Let us consider a term to be added to a $J^{(4)}$ current

$$\partial U = c_{ab}\partial(\partial^2\phi_a\partial\phi_b) \tag{2.23}$$

We start by evaluating $\bar\partial\langle\partial U\rangle$ using the massless propagator in (2.13) and dropping all non–local contributions (with the definition $\alpha \equiv \frac{\beta^2}{2\pi}$)

$$\begin{aligned}
&\bar\partial\left\langle\partial(\partial^2\phi_a\partial\phi_b)\right\rangle = \partial\bar\partial\left\langle(\partial^2\phi_a\partial\phi_b)e^{-\frac{1}{2\pi\alpha}\int d^2w\ V}\right\rangle_0 \sim \\
&\sim \partial\left\{-\frac{1}{2\pi\alpha}\int d^2w\left[\frac{\alpha}{2}\partial_x\phi_b\ \bar\partial\left(\frac{1}{(x-w)^2}+\frac{1}{(x-\bar w)^2}\right)V_a(w,\bar w)+\right.\right.\\
&\qquad -\frac{\alpha}{2}\partial_x^2\phi_a\ \bar\partial\left(\frac{1}{x-w}+\frac{1}{x-\bar w}\right)V_b(w,\bar w)+\\
&\qquad \left.\left.-\frac{\alpha^2}{4}\bar\partial\left(\frac{1}{(x-w)^3}+\frac{1}{(x-\bar w)^3}\right)V_{ab}(w,\bar w)\right]\right\}\sim \\
&\sim \partial\left[\frac{1}{2}\partial V_a\partial\phi_b+\frac{1}{2}V_b\partial^2\phi_a+\frac{\alpha}{8}\partial^2V_{ab}\right]
\end{aligned} \tag{2.24}$$

Here and in the following we write derivatives of the interactions as $V_a \equiv \frac{\partial V}{\partial\phi_a}$, $B_a \equiv \frac{\partial B}{\partial\phi_a}$, and so on. The contribution to the trace (see eq.(2.14)) is then identified as

$$\Theta \to -c_{ab}\left[\frac{1}{2}\partial V_a\partial\phi_b+\frac{1}{2}V_b\partial^2\phi_a+\frac{\alpha}{8}\partial^2V_{ab}\right] \tag{2.25}$$

In the same way one obtains the corresponding contributions to $\tilde J$ and $\tilde\Theta$ so that one can compute the relevant terms produced at the boundary from $\langle J^{(1)}\rangle$ as in (2.15)

$$\begin{aligned}
\left\langle J^{(1)}\right\rangle \;=\; & i\left\langle J-\tilde J-\Theta+\tilde\Theta\right\rangle \to ic_{ab}\left\langle\left[\partial^3\phi_a\partial\phi_b+\partial^2\phi_a\partial^2\phi_b+\frac{1}{2}\partial V_a\partial\phi_b+\right.\right.\\
&\left.\left.+\frac{1}{2}V_b\partial^2\phi_a+\frac{\alpha}{8}\partial^2V_{ab}-c.c.\right]e^{-\frac{1}{2\pi\alpha}\int d^2w\ V+\frac{1}{2\pi\alpha}\int dw_0\ B}\right\rangle_0
\end{aligned} \tag{2.26}$$

One needs consider terms up to the third–order expansion in B and to second order in the V and B crossed product. We list here the results from the individual terms and compute in detail the first one in Appendix A.

$$\begin{aligned}
i\left\langle\partial^3\phi_a\partial\phi_b-c.c.\right\rangle\Big|_{x_1=0} &\sim -2B_b\partial_0^3\phi_a-2\partial_0^2B_a\partial_0\phi_b+\frac{3}{2}B_b\partial_0V_a+\frac{1}{2}V_{ac}B_c\partial_0\phi_b\\
i\left\langle\partial^2\phi_a\partial^2\phi_b-c.c.\right\rangle\Big|_{x_1=0} &\sim -2\partial_0B_a\partial_0^2\phi_b-2\partial_0B_b\partial_0^2\phi_a+V_a\partial_0B_b+V_b\partial_0B_a-\frac{2}{3}\,\alpha\partial_0^3B_{ab}\\
\frac{i}{2}\left\langle\partial V_a\partial\phi_b-c.c.\right\rangle\Big|_{x_1=0} &\sim -\frac{1}{2}B_b\partial_0V_a-\frac{1}{2}V_{ac}B_c\partial_0\phi_b-\alpha V_{ac}\partial_0B_{bc}
\end{aligned}$$

$$
\begin{aligned}
\frac{i}{2}\left\langle V_b\partial^2\phi_a - c.c.\right\rangle\Big|_{x_1=0} &\sim -V_b\partial_0 B_a \\
\frac{i\alpha}{8}\left\langle \partial^2 V_{ab} - c.c.\right\rangle\Big|_{x_1=0} &\sim -\frac{\alpha}{4}\left[\partial_0(V_{abc}B_c) + \alpha V_{abcd}\partial_0 B_{cd}\right]
\end{aligned}
\tag{2.27}
$$

The total sum finally gives

$$
\begin{aligned}
\langle J^{(1)}\rangle|_{x_1=0} \sim\ & c_{ab}\left\{\partial_0\left[-2\partial_0 B_a\partial_0\phi_b - 2B_b\partial_0^2\phi_a - \frac{2}{3}\alpha\partial_0^2 B_{ab}\right] + \partial_0\left(V_a B_b\right) + \right. \\
& \left. -\alpha\left[V_{ac}\partial_0 B_{bc} + \frac{1}{4}\partial_0(V_{abc}B_c) + \frac{\alpha}{4}V_{abcd}\partial_0 B_{cd}\right]\right\}
\end{aligned}
\tag{2.28}
$$

We notice that contributions containing three ∂_0 derivatives, both classical and quantum, add up to reconstruct a total ∂_0 derivative. Terms containing one ∂_0 derivative behave differently depending on whether they were produced at tree level or from loops: as expected the classical terms give rise to a total ∂_0–derivative contribution. The terms instead which correspond to quantum corrections can modify the boundary condition in a nontrivial manner and they *must* be taken into account while constructing the quantum conserved charges. A final remark: total derivatives of the form $c_a\partial^n\phi_a$ are not relevant since they would only contribute at the classical level; we will not consider them in the following.

We turn now to a discussion of the conservation laws for the specific cases of the spin–3 and spin–4 currents in Toda theories defined in the upper–half plane, perturbed by a boundary interaction.

2.1 Quantum conservation at spin–3 level

For a Toda theory the action in (2.3) is written in terms of n independent scalar fields interacting with the potential (2.4) in the inner region and with a generic perturbation B at the boundary. According to the general expression in (2.16) we write a spin–3 current in the form

$$
J^{(3)} = \frac{1}{3}a_{abc}\partial\phi_a\partial\phi_b\partial\phi_c + b_{ab}\partial^2\phi_a\partial\phi_b + \frac{1}{2}c_{ab}\partial(\partial\phi_a\partial\phi_b)
\tag{2.29}
$$

with coefficients a_{abc} and c_{ab} symmetric and b_{ab} antisymmetric in their indices. We start considering the conservation law of this current in the upper–half plane following the procedure outlined for the general case. We evaluate as in (2.14) $\bar\partial\left\langle J^{(3)}\right\rangle$. We easily find

$$
\begin{aligned}
\bar\partial\left\langle J^{(3)}\right\rangle &= \bar\partial\left\langle J^{(3)} e^{-\frac{1}{2\pi\alpha}\int d^2w\ V}\right\rangle_0 \sim \bar\partial\left\langle J^{(3)}\left(-\frac{1}{2\pi\alpha}\right)\int d^2w\ V\right\rangle_0 \sim \\
&\sim \partial\left[\frac{1}{2}b_{ab}V_b\partial\phi_a + \frac{1}{2}c_{ab}V_b\partial\phi_a + \frac{\alpha}{8}c_{ab}\partial V_{ab} + \frac{\alpha^2}{48}a_{abc}\partial V_{abc}\right] + \\
&+ \frac{1}{2}\left[a_{abc}V_a + 2b_{ac}V_{ab} + \frac{\alpha}{2}a_{abd}V_{acd}\right]\partial\phi_b\partial\phi_c
\end{aligned}
\tag{2.30}
$$

where we have dropped all the non–local contributions. Absence of quantum anomalies in the conservation of $J^{(3)}$ requires that the terms on the right–hand–side, which are not total ∂–derivatives, vanish. This requirement leads to the following equations for the a_{abc} and b_{ab} coefficients

$$a_{abc}V_a + b_{ac}V_{ab} + b_{ab}V_{ac} + \frac{\alpha}{4}a_{abd}V_{acd} + \frac{\alpha}{4}a_{acd}V_{abd} = 0 \tag{2.31}$$

Clearly no restrictions are imposed at this stage on the coefficients c_{ab} of the total derivative terms.

From equation (2.30) we also determine the quantum trace

$$\Theta^{(3)} = -\frac{1}{2}b_{ab}V_b\partial\phi_a - \frac{1}{2}c_{ab}V_b\partial\phi_a - \frac{\alpha}{8}c_{ab}\partial V_{ab} - \frac{\alpha^2}{48}a_{abc}\partial V_{abc} \tag{2.32}$$

The same procedure can be applied to compute the quantum currents $\tilde{J}^{(3)}$, $\tilde{\Theta}^{(3)}$ whose expressions are obtained from (2.29), (2.32) by exchanging holomorphic derivatives with antiholomorphic ones.

Now we concentrate on the boundary condition (2.15). Thus we consider

$$\left\langle i\left(J^{(3)} - \tilde{J}^{(3)} - \Theta^{(3)} + \tilde{\Theta}^{(3)}\right) e^{-\frac{1}{2\pi\alpha}\int d^2w \; V + \frac{1}{2\pi\alpha}\int dw_0 \; B}\right\rangle_0\Big|_{x_1=0} \tag{2.33}$$

Local corrections come from contractions of the currents with the exponential expanded up to the third order. Summing all the contributions the final result, up to total ∂_0 derivatives, is

$$\begin{aligned}
&\left\langle i(J^{(3)} - \tilde{J}^{(3)} - \Theta^{(3)} + \tilde{\Theta}^{(3)})\right\rangle\Big|_{x_1=0} \sim \\
&\sim \frac{1}{\sqrt{2}}\left[\frac{1}{3}a_{abc}B_aB_bB_c + 2b_{ab}V_aB_b - \frac{\alpha}{4}c_{ab}V_{abc}B_c - \frac{\alpha^2}{24}a_{abc}V_{abcd}B_d\right] + \\
&-\frac{1}{\sqrt{2}}\left[a_{abc}B_a + 4b_{ab}B_{ac} + 2\alpha a_{abd}B_{acd}\right]\partial_0\phi_b\partial_0\phi_c
\end{aligned} \tag{2.34}$$

In order to cancel the terms proportional to $\partial_0\phi_b\partial_0\phi_c$ we require

$$a_{abc}B_a + 2b_{ab}B_{ac} + 2b_{ac}B_{ab} + \alpha a_{acd}B_{abd} + \alpha a_{abd}B_{acd} = 0 \tag{2.35}$$

Comparing (2.35) with (2.31) it is easy to see that the quantum corrections in both identities are such that if (2.31) is satisfied with $V = \sum_{j=0}^{n} q_j e^{\vec{\alpha}_j\cdot\vec{\phi}}$, then (2.35) is also satisfied with $B = \sum_{j=0}^{n} d_j e^{\frac{1}{2}\vec{\alpha}_j\cdot\vec{\phi}}$. Finally, once we have found the a_{abc}'s and b_{ab}'s from eq.(2.31) or equivalently from (2.35), we try to determine the coefficients c_{ab} in the current and the d_j's in the boundary interaction imposing (see again (2.34))

$$\frac{1}{3}a_{abc}B_aB_bB_c + 2b_{ab}V_aB_b - \frac{\alpha}{4}c_{ab}V_{abc}B_c - \frac{\alpha^2}{24}a_{abc}V_{abcd}B_d = 0 \tag{2.36}$$

If we are able to satisfy this condition then we can proceed and construct the corresponding quantum conserved charge. In section 3 we explicitly solve the above equations for the $a_n^{(1)}$ affine Toda theories.

2.2 Quantum conservation at spin–4 level

The most general expression for the current of spin four, including total derivative contributions, has the form

$$\begin{aligned} J^{(4)} &= \frac{1}{4}a_{abcd}\partial\phi_a\partial\phi_b\partial\phi_c\partial\phi_d + \frac{1}{2}b_{abc}\partial^2\phi_a\partial\phi_b\partial\phi_c + \frac{1}{3}c_{abc}\partial(\partial\phi_a\partial\phi_b\partial\phi_c) + \\ &\quad + \frac{1}{2}d_{ab}\partial^2\phi_a\partial^2\phi_b + e_{ab}\partial(\partial^2\phi_a\partial\phi_b) \end{aligned} \tag{2.37}$$

with a_{abcd}, c_{abc} and d_{ab} completely symmetric, b_{abc} symmetric in the last two indices and $b_{abc} + b_{cab} + b_{bca} = 0$. The requirement of current conservation in the interior region (see eq.(2.14)) determines the quantum trace

$$\begin{aligned} \Theta^{(4)} &= -\Big[\frac{1}{4}b_{bca}V_a\partial\phi_b\partial\phi_c + \frac{1}{2}c_{abc}V_a\partial\phi_b\partial\phi_c + \frac{1}{4}d_{ab}\partial V_a\partial\phi_b + \frac{1}{2}e_{ab}\partial V_a\partial\phi_b + \\ &\quad + \frac{1}{2}e_{ab}V_b\partial^2\phi_a + \frac{\alpha}{8}b_{(ab)c}\partial V_{ac}\partial\phi_b + \frac{\alpha}{4}c_{abc}\partial V_{ab}\partial\phi_c + \frac{\alpha}{48}d_{ab}\partial^2 V_{ab} + \frac{\alpha}{8}e_{ab}\partial^2 V_{ab} + \\ &\quad + \frac{\alpha^2}{32}a_{abcd}\partial V_{abc}\partial\phi_d + \frac{\alpha^2}{48}c_{abc}\partial^2 V_{abc} + \frac{\alpha^3}{384}a_{abcd}\partial^2 V_{abcd}\Big] \end{aligned} \tag{2.38}$$

and imposes the following two sets of conditions on the coefficients of the current

$$\begin{aligned} & a_{a(bcd)}V_a + \frac{1}{2}b_{a(bc}V_{d)a} - \frac{1}{2}V_{a(d}b_{bc)a} - \frac{1}{2}d_{a(b}V_{cd)a} + \frac{3}{4}\alpha a_{ae(cd}V_{b)ae} + \\ &\quad + \frac{\alpha}{8}b_{ae(b}V_{cd)ae} - \frac{\alpha}{8}V_{ae(cd}b_{b)ae} + \frac{\alpha^2}{16}a_{aef(d}V_{bc)aef} = 0 \end{aligned}$$

$$b_{[bc]a}V_a + d_{a[b}V_{c]a} + \frac{\alpha}{4}(b_{[ba]e}V_{ace} - b_{[ca]e}V_{abe}) + \frac{\alpha^2}{8}a_{aef[c}V_{b]aef} = 0 \tag{2.39}$$

Then we consider the boundary relation (2.15). In this case we obtain three sets of equations which the coefficients and the boundary potential need satisfy in order to insure the existence of a corresponding quantum charge. The first two sets arise from terms proportional to $\partial_0\phi_b\partial_0\phi_c\partial_0\phi_d$ and $\partial_0^2\phi_b\partial_0\phi_c$ respectively

$$\begin{aligned} & a_{a(bcd)}B_a + b_{a(bc}B_{d)a} - B_{a(d}b_{bc)a} - 2d_{a(b}B_{cd)a} + 3\alpha a_{ae(cd}B_{b)ae} + \alpha b_{ae(b}B_{cd)ae} + \\ &\quad - \alpha B_{ae(cd}b_{b)ae} + \alpha^2 a_{aef(d}B_{bc)aef} = 0 \end{aligned}$$

$$b_{[bc]a}B_a + 2d_{a[b}B_{c]a} + \alpha(b_{[ba]e}B_{ace} - b_{[ca]e}B_{abe}) + \alpha^2 a_{aef[c}B_{b]aef} = 0 \tag{2.40}$$

In a way similar to what happened for the spin–3 current, the above equations do not impose new conditions once the bulk equations (2.39) are satisfied and the boundary potential is of the form $B = \sum_{j=0}^{n} d_j e^{\frac{1}{2}\vec{\alpha}_j\cdot\vec{\phi}}$. The relevant equations are given instead by the terms which are proportional to $\partial_0\phi_e$; they must reduce to a total ∂_0–derivative in order to satisfy (2.15). We find

$$
\begin{aligned}
&\Big[a_{abce}B_aB_bB_c + b_{abc}B_bB_cB_{ae} + 3\alpha a_{abcd}B_bB_cB_{ade} + \\
&+ b_{aec}V_aB_c - \frac{1}{2}b_{bea}V_aB_b - \frac{1}{2}b_{eca}V_aB_c + 2d_{ab}V_aB_{be} - d_{a(b}V_{e)a}B_b + \alpha b_{abc}V_aB_{bce} + \\
&- \alpha b_{bca}V_aB_{bce} - \frac{\alpha}{4}b_{(ab)c}V_{ace}B_b - \frac{\alpha}{4}b_{(ae)c}V_{abc}B_b - \alpha c_{ab(c}V_{e)ab}B_c + \\
&- \frac{\alpha}{12}(d_{ab}+6e_{ab})V_{abce}B_c - \frac{\alpha}{12}(d_{ab}+6e_{ab})V_{abc}B_{ec} - \alpha(d_{a(b}V_{c)a} + 2e_{a(b}V_{c)a})B_{bce} + \\
&- \frac{\alpha^2}{8}a_{acd(e}V_{b)acd}B_b - \frac{\alpha^2}{2}b_{(ab)d}V_{acd}B_{bce} - \alpha^2 c_{abc}V_{abd}B_{cde} + \\
&- \frac{\alpha^2}{12}(d_{ab}+6e_{ab})V_{abcd}B_{cde} - \frac{\alpha^3}{96}a_{abcd}V_{abcdef}B_f - \frac{\alpha^3}{96}a_{abcd}V_{abcdf}B_{ef} + \\
&- \frac{\alpha^3}{8}a_{adf(c}V_{b)adf}B_{bce} - \frac{\alpha^3}{12}c_{abc}V_{abcdf}B_{def} - \frac{\alpha^4}{96}a_{abcd}V_{abcdfg}B_{efg}\Big]\,\partial_0\phi_e \\
&\sim \partial_0 - derivative
\end{aligned}
\tag{2.41}
$$

In the following sections we attempt to find explicit solutions of the above equations for specific models.

3 The $a_n^{(1)}$ affine Toda theories

The action for these simply laced theories has the general form (2.3), (2.4), with n independent fields and roots satisfying $\vec{\alpha}_j^2 = 2$, $\vec{\alpha}_j \cdot \vec{\alpha}_k = -\delta_{j,k\pm1}$, $j,k = 1,\cdots n$, and $q_j = 1$, $j = 1,\cdots,n$.

These models possess a classically conserved spin–3 current of the form considered in (2.29) [15]. It has also been established [4] that the conservation law in the inner region is valid at the quantum level. If we call $a_{abc}^{(0)}$ and $b_{ab}^{(0)}$ the classical coefficients, the quantum solution is given by

$$
a_{abc} = a_{abc}^{(0)} \qquad , \qquad b_{ab} = (1+\frac{\alpha}{2})b_{ab}^{(0)} \tag{3.1}
$$

At spin 3 the boundary condition (2.15) corresponds to equations (2.35) and (2.36). The first set, (2.35), is satisfied by the coefficients in (3.1) if the boundary potential is chosen as in (2.10). Now, in order to satisfy (2.36) which is nonlinear in B, we still have

the freedom to choose appropriately the coefficients c_{ab} in the current and the d_j's in the interaction B. It is convenient to introduce the following definitions

$$a_{ijk} \equiv a_{abc}(\alpha_i)_a(\alpha_j)_b(\alpha_k)_c \qquad b_{ij} \equiv b_{ab}(\alpha_i)_a(\alpha_j)_b \qquad c_{ij} \equiv c_{ab}(\alpha_i)_a(\alpha_j)_b \tag{3.2}$$

so that from (2.31) we first obtain

$$a_{ijk} + b_{ik}C_{ij} + b_{ij}C_{ik} + \frac{\alpha}{4}[a_{iij}C_{ik} + a_{iik}C_{ij}] = 0 \tag{3.3}$$

where $C_{ij} \equiv \vec{\alpha}_i \cdot \vec{\alpha}_j$ is the Cartan matrix of the a_n Lie algebra. From the above equation and the antisymmetry of b_{ij} one easily derives

$$\begin{aligned} a_{iii} &= 0 \\ a_{iij} &= -a_{jji} = -\frac{2}{1+\frac{\alpha}{2}}b_{ij} \end{aligned} \tag{3.4}$$

for any $i,j = 0,1,\cdots,n$. Finally introducing the notation

$$V = \sum_{j=0}^{n} e_j^2 \qquad\qquad B = \sum_{j=0}^{n} d_j e_j \tag{3.5}$$

where

$$e_j \equiv e^{\frac{1}{2}\vec{\alpha}_j\cdot\vec{\phi}} \tag{3.6}$$

we rewrite (2.36) as

$$\sum_{i\neq j}\left(-\frac{1}{4(1+\frac{\alpha}{2})}d_i^2 + 1\right) b_{ij}d_j e_i^2 e_j - \frac{\alpha}{8}\sum_{i,j} c_{ii}C_{ij}d_j e_i^2 e_j = 0 \tag{3.7}$$

It is clear that when in the second sum $i = j$ terms proportional to e_i^3 are produced and the only way to cancel them is to impose

$$c_{ii} = 0 \tag{3.8}$$

When $i \neq j$ the coefficients c_{ij} do not enter in (3.7), so they are undetermined and not relevant. Consequently, in order to satisfy (3.7) the coefficients d_j must be chosen as

$$d_j^2 = 4\left(1+\frac{\alpha}{2}\right) \quad , \qquad j = 0,1,\cdots,n \tag{3.9}$$

We note that setting $\alpha = 0$ the classical result in Ref. [9, 10] is reproduced, with $d_j^2 = 4$, $j = 0,1,\cdots,n$. However the presence of quantum corrections modifies this solution: the conservation of the $q^{(2)}$ charge requires a nonperturbative, finite renormalization of the coefficients d_j.

Now we extend the analysis at spin 4. The general equations have been obtained in the previous section, (2.37), (2.38), (2.39), (2.41). We have not been able to exhibit a solution in the quantum case for a generic $a_n^{(1)}$ Toda system, essentially because the various sets of equations become highly coupled due to the presence of the perturbative corrections. So we present the results we have obtained for the explicit cases $n = 3, 4, 5$ (a classical spin–4 conserved current exists only for $n \geq 3$). Here we discuss in detail the $n = 3$ example.

The $a_3^{(1)}$ theory is described by three independent scalar fields. The simple roots can be represented in the following real form

$$\vec{\alpha}_1 = (-1, -1, 0) \qquad \vec{\alpha}_2 = (1, 0, 1) \qquad \vec{\alpha}_3 = (-1, 1, 0) \tag{3.10}$$

so that the bulk potential (2.4) becomes

$$V = e^{\phi_1 - \phi_3} + e^{-\phi_1 - \phi_2} + e^{\phi_1 + \phi_3} + e^{-\phi_1 + \phi_2} \tag{3.11}$$

and the interaction at the boundary (2.10) is

$$B = d_0 e^{\frac{1}{2}(\phi_1 - \phi_3)} + d_1 e^{\frac{1}{2}(-\phi_1 - \phi_2)} + d_2 e^{\frac{1}{2}(\phi_1 + \phi_3)} + d_3 e^{\frac{1}{2}(-\phi_1 + \phi_2)} \tag{3.12}$$

The lagrangian is clearly symmetric under

$$\begin{array}{lll} i) \quad \phi_2 \to -\phi_2 & , & ii) \quad \phi_3 \to -\phi_3, \\ iii) \quad \phi_1 \to -\phi_1 & & \phi_2 \to \phi_3 \end{array} \tag{3.13}$$

In this case it is rather easy to solve the equations (2.39) and find the coefficients a_{abc}, b_{abc} and d_{ab} which appear in (2.37) and (2.38). The coefficients c_{abc} and e_{ab} are not determined from the equations in the interior region. Based on the symmetries in (3.13) the only nonvanishing ones are

$$c_{122} = -c_{133} \equiv G \qquad e_{11} \equiv H \qquad e_{22} = e_{33} \equiv I \tag{3.14}$$

We have

$$\begin{aligned} J^{(4)} \ = \ & (\partial\phi_1)^4 + (\partial\phi_2)^4 + (\partial\phi_3)^4 - 6\left[(\partial\phi_1)^2(\partial\phi_2)^2 + (\partial\phi_1)^2(\partial\phi_3)^2 + (\partial\phi_2)^2(\partial\phi_3)^2\right] + \\ & - 24(1 + \frac{\alpha}{2})\left[\partial^2\phi_2\partial\phi_2\partial\phi_1 - \partial^2\phi_3\partial\phi_3\partial\phi_1\right] + (4 + \frac{\alpha^2}{4})(\partial^2\phi_1)^2 + \\ & - (8 + 12\alpha + \frac{11}{4}\alpha^2)[(\partial^2\phi_2)^2 + (\partial^2\phi_3)^2] + G\partial\left(\partial\phi_1(\partial\phi_2)^2 - \partial\phi_1(\partial\phi_3)^2\right) + \\ & + H\partial(\partial\phi_1\partial^2\phi_1) + I\left[\partial(\partial\phi_2\partial^2\phi_2) + \partial(\partial\phi_3\partial^2\phi_3)\right] \end{aligned} \tag{3.15}$$

and

$$\Theta^{(4)} = \Theta_1 + \Theta_2 + \Theta_3 + \Theta_0 \tag{3.16}$$

where

$$\begin{aligned}
\Theta_1 &= \Bigg\{ -\left(2 + \frac{17}{6}\alpha + \alpha^2 + \frac{5}{48}\alpha^3 - \frac{\alpha^2}{16}G + \frac{\alpha^2}{8}(H+I)\right)(\partial\phi_1 + \partial\phi_2)^2 + \\
&\quad +6\left(1 + \frac{\alpha}{2} - \frac{1}{12}G\right)(\partial\phi_3)^2 + \left(-\frac{1}{2}H + \frac{\alpha}{4}G\right)(\partial\phi_1)^2 + \left(\frac{1}{2}(1+\alpha)G - \frac{1}{2}I\right)(\partial\phi_2)^2 + \\
&\quad + \left((1 + \frac{3}{4}\alpha)G - \frac{1}{2}(H+I)\right)\partial\phi_1\partial\phi_2 + \frac{1}{2}H\partial^2\phi_1 + \frac{1}{2}I\partial^2\phi_2 + \\
&\qquad + \left(\frac{17}{6}\alpha + \frac{3}{4}\alpha^2 + \frac{5}{48}\alpha^3 - \frac{\alpha^2}{16}G + \frac{\alpha^2}{8}(H+I)\right)(\partial^2\phi_1 + \partial^2\phi_2)\Bigg\} e^{-\phi_1-\phi_2} \\
\Theta_2 &= \Theta_1(\phi_1 \to -\phi_1, \phi_2 \to -\phi_3) \\
\Theta_3 &= \Theta_1(\phi_2 \to -\phi_2) \\
\Theta_0 &= \Theta_1(\phi_1 \to -\phi_1, \phi_2 \to \phi_3)
\end{aligned} \tag{3.17}$$

Now we attempt to solve the conditions at the boundary. As previously emphasized the relevant set of equations are the ones in (2.41). They are non linear in B, with terms cubic in B and terms which contain the product BV. Using the definitions in (3.6) let us isolate for example terms proportional to $e_0e_1e_2$, which arise only from contributions cubic in B (they are not contained in the products BV). Therefore of all the terms in (2.41) we concentrate on the first three and from them we extract the part proportional to $e_0e_1e_2$. We easily find that the classical contributions cancel and we are left with

$$-\frac{3}{4}\alpha(\partial_0\phi_1 + \partial_0\phi_2)d_0d_1d_2e_0e_1e_2 \equiv -\frac{3}{4}\alpha(\partial_0\phi_1 + \partial_0\phi_2)d_0d_1d_2e^{\frac{1}{2}(\phi_1-\phi_2)} \tag{3.18}$$

This explicitly shows that there is no way to rewrite this type of contributions as total ∂_0 derivatives unless we set one of the $d_j, j = 0,1,2,$ coefficients equal to zero. On the other hand since the theory is symmetric under (3.13) the same analysis can be repeated for the terms proportional to $e_1e_2e_3$, $e_2e_3e_0$, $e_3e_0e_1$. Thus one is forced to set at least two of the d_j's equal to zero. In this case it is rather simple to show that the other nontrivial conditions which follow from (2.41) necessarily require the vanishing of the remaining two d_j coefficients. As expected the original symmetry of the lagrangian is maintained. In conclusion the conservation at the boundary can be implemented only for a vanishing interaction at the border.

We have repeated the corresponding analysis for the $a_4^{(1)}$ and the $a_5^{(1)}$ Toda theories. These models are described by four and five scalar fields, respectively. In order to simplify the algebra we have found convenient to use a realization of the simple roots of a_n

that maintains explicit all the symmetries of the corresponding affine Dynkin diagram. This can be achieved choosing a complex representation of the roots as in Ref. [2]. Moreover with that particular choice, as shown in Ref. [2], the reality of the lagrangian is implemented representing the fields in a complex basis with $\phi_a^* = \phi_{n+1-a}$. It is a simple exercise to modify accordingly the equations (2.41) which are the relevant ones for checking the conservation at the boundary. For the specific examples of the $a_4^{(1)}$ and the $a_5^{(1)}$ Toda systems we have performed most of the algebraic manipulations using Mathematica. In both cases we have found that the classical conservation of the $q^{(3)}$ charge is broken by quantum anomalous contributions, following exactly the same pattern as for the $a_3^{(1)}$ theory. It is from the terms cubic in B that in (2.41) arise contributions which do not sum up to a total ∂_0 derivative. There is no choice of the coefficients c_{abc} and e_{ab}, still undetermined in the $J^{(4)}$ and $\Theta^{(4)}$ currents, which allows to satisfy the conservation condition with a nonvanishing boundary potential. We suspect that a similar situation has to be faced at higher spin levels.

4 Conservation laws for nonsimply laced theories

In this section we study the conservation equations of the first nontrivial higher–spin current, spin 4, for the two nonsimply laced $d_3^{(2)}$ and $c_2^{(1)}$ theories. These models, described in terms of two bosonic fields, are simple enough to allow a complete analysis. We discuss the two systems separately.

4.1 The $d_3^{(2)}$ system

With a realization of the simple roots of the Lie algebra as

$$\vec{\alpha}_1 = (2,0) \qquad \vec{\alpha}_2 = (-1,1) \tag{4.1}$$

one obtains for the potential in the bulk

$$V = e^{-\phi_1-\phi_2} + e^{2\phi_1} + e^{-\phi_1+\phi_2} \tag{4.2}$$

and at the boundary

$$B = d_0\, e^{-\frac{1}{2}(\phi_1+\phi_2)} + d_1\, e^{\phi_1} + d_2\, e^{-\frac{1}{2}(\phi_1-\phi_2)} \tag{4.3}$$

This system exhibits the first high–spin conserved current at spin–4, with a general form given in (2.37). Solving the bulk conservation equations (2.39) one finds the coefficients

a_{abcd}, b_{abc} and d_{ab}; the nonvanishing ones are [4]

$$\begin{aligned} a_{1111} &= a_{2222} = -\frac{\alpha}{3} \qquad\qquad a_{1122} = \frac{2}{3}(1+\frac{\alpha}{2}) \qquad\qquad b_{221} = 2+3\alpha+\alpha^2 \\ d_{11} &= -\frac{\alpha}{6}(1+3\alpha+\alpha^2) \qquad\qquad d_{22} = 2(1+\frac{23}{12}\alpha+\alpha^2+\frac{\alpha^3}{6}) \end{aligned} \tag{4.4}$$

The coefficients of the terms which are total derivatives are not determined; the ones which are allowed by the symmetry of the lagrangian under $\phi_2 \to -\phi_2$ are

$$c_{122} \equiv G \qquad c_{111} \equiv 3H \qquad e_{11} \equiv I \qquad e_{22} \equiv J \tag{4.5}$$

Inserting (4.4) in the general expression (2.38) for the quantum trace we obtain

$$\Theta^{(4)} = \Theta_0 + \Theta_1 + \Theta_2 \tag{4.6}$$

where

$$\begin{aligned} \Theta_0 &= \Bigg\{\left[-\frac{\alpha}{8}\left(1+\frac{31}{18}\alpha+\frac{5}{6}\alpha^2+\frac{1}{9}\alpha^3\right)+\frac{\alpha^2}{16}(G+H)-\frac{\alpha}{8}(I+J)\right](\partial\phi_1+\partial\phi_2)^2 + \\ &\quad + \left[\frac{\alpha}{4}G+\frac{3}{2}\left(1+\frac{\alpha}{2}\right)H-\frac{1}{2}I\right](\partial\phi_1)^2 + \\ &\quad + \left[\frac{1}{2}(1+\alpha)G-\frac{1}{2}J\right](\partial\phi_2)^2 + \\ &\quad + \left[\left(1+\frac{3}{4}\alpha\right)G+\frac{3}{4}\alpha H-\frac{1}{2}(I+J)\right]\partial\phi_1\partial\phi_2 + \\ &\quad + \left[-\frac{\alpha}{24}\left(1+\frac{5}{6}\alpha-\frac{\alpha^2}{2}-\frac{\alpha^3}{3}\right)-\frac{\alpha^2}{16}(G+H)+\frac{\alpha}{8}(I+J)\right](\partial^2\phi_1+\partial^2\phi_2) + \\ &\quad + \frac{1}{2}I\ \partial^2\phi_1+\frac{1}{2}J\ \partial^2\phi_2\Bigg\}\, e^{-\phi_1-\phi_2} \\ \Theta_1 &= \Bigg\{\left[-\frac{\alpha}{6}\left(1+\frac{10}{3}\alpha+3\alpha^2+\frac{2}{3}\alpha^3\right)-(3+6\alpha+2\alpha^2)H-2(1+\alpha)I\right](\partial\phi_1)^2 + \\ &\quad + \left(1+\frac{3}{2}\alpha+\frac{1}{2}\alpha^2-G\right)(\partial\phi_2)^2 + \\ &\quad + \left[-\frac{\alpha^2}{36}(1+3\alpha+2\alpha^2)-\alpha^2 H-(1+\alpha)I\right]\partial^2\phi_1\Bigg\}\, e^{2\phi_1} \\ \Theta_2 &= \Theta_0(\phi_2\to-\phi_2) \end{aligned} \tag{4.7}$$

The requirement of absence of anomalies at the boundary (2.15) leads to the set of equations in (2.41). In the specific case under consideration they give

$$\frac{\alpha}{2}(G-3H)d_0 = 0$$

$$
\begin{aligned}
&(3+2\alpha-\alpha^2)d_1^2 d_0 - \Big[6+10\alpha+\frac{8}{3}\alpha^2-\frac{13}{9}\alpha^3-\alpha^4-\frac{2}{9}\alpha^5+ \\
&\qquad\qquad +4\alpha(3+3\alpha+\alpha^2)H+4\alpha(1+\alpha)I\Big]\,d_0=0 \\
&\frac{\alpha}{4}d_0^2 d_1+\frac{\alpha}{2}\Big[1+\frac{5}{2}\alpha+\frac{55}{36}\alpha^2+\frac{5}{12}\alpha^3+\frac{\alpha^4}{18}+(1+\alpha+\frac{\alpha^2}{4})G+ \\
&\qquad\qquad +(3+3\alpha+\frac{\alpha^2}{4})H-(2+\frac{1}{2}\alpha)I-\frac{1}{2}\alpha J\Big]\,d_1=0 \\
&\frac{\alpha}{2}(G-3H)d_2=0 \\
&(3+2\alpha-\alpha^2)d_1^2 d_2 - \Big[6+10\alpha+\frac{8}{3}\alpha^2-\frac{13}{9}\alpha^3-\alpha^4-\frac{2}{9}\alpha^5+ \\
&\qquad\qquad +4\alpha(3+3\alpha+\alpha^2)H+4\alpha(1+\alpha)I\Big]\,d_2=0 \\
&\frac{\alpha}{4}d_2^2 d_1+\frac{\alpha}{2}\Big[1+\frac{5}{2}\alpha+\frac{55}{36}\alpha^2+\frac{5}{12}\alpha^3+\frac{\alpha^4}{18}+(1+\alpha+\frac{\alpha^2}{4})G+ \\
&\qquad\qquad +(3+3\alpha+\frac{\alpha^2}{4})H-(2+\frac{1}{2}\alpha)I-\frac{1}{2}\alpha J\Big]\,d_1=0 \qquad (4.8)
\end{aligned}
$$

In the limit $\alpha \to 0$ all the terms which contain G, H, I, J vanish and one recovers the classical boundary equations whose solution fixes the coefficients d_j's

$$
\begin{aligned}
&a) \quad d_0=d_2=0 \qquad\qquad d_1 \quad arbitrary \\
&b) \quad d_1=\pm\sqrt{2} \qquad\quad d_0,\ d_2 \quad arbitrary \qquad (4.9)
\end{aligned}
$$

This result is in agreement with Ref. [10].

Now we analyze the equations in (4.8) at the quantum level. First we try to find a solution setting to zero all the terms which are proportional to total derivatives in the current, i.e. setting $G=H=I=J=0$. The system in (4.8) reduces to

$$
\begin{aligned}
&(3+2\alpha-\alpha^2)d_1^2 d_0 - \Big[6+10\alpha+\frac{8}{3}\alpha^2-\frac{13}{9}\alpha^3-\alpha^4-\frac{2}{9}\alpha^5\Big]\,d_0=0 \\
&\frac{\alpha}{4}\Big[d_0^2 d_1+\Big(2+5\alpha+\frac{55}{18}\alpha^2+\frac{5}{6}\alpha^3+\frac{\alpha^4}{9}\Big)\,d_1\Big]=0 \\
&(3+2\alpha-\alpha^2)d_1^2 d_2 - \Big[6+10\alpha+\frac{8}{3}\alpha^2-\frac{13}{9}\alpha^3-\alpha^4-\frac{2}{9}\alpha^5\Big]\,d_2=0 \\
&\frac{\alpha}{4}\Big[d_2^2 d_1+\Big(2+5\alpha+\frac{55}{18}\alpha^2+\frac{5}{6}\alpha^3+\frac{\alpha^4}{9}\Big)\,d_1\Big]=0 \qquad (4.10)
\end{aligned}
$$

These equations give either the trivial boundary solution $d_0=d_1=d_2=0$ or

$$
\begin{aligned}
&d_1^2=2+\frac{\frac{2}{9}\alpha^5+\alpha^4+\frac{13}{9}\alpha^3-\frac{14}{3}\alpha^2-6\alpha}{\alpha^2-2\alpha-3} \\
&d_0^2=d_2^2=-\Big[2+5\alpha+\frac{55}{18}\alpha^2+\frac{5}{6}\alpha^3+\frac{\alpha^4}{9}\Big] \qquad (4.11)
\end{aligned}
$$

Clearly these solutions are not acceptable, primarily because the d_0 and d_2 coefficients have imaginary values and therefore the theory does not appear to be unitary. Thus we are forced to reconsider the original system in (4.8) with nonvanishing constants G, H, I and J. In this case the solution is not unique. We use this freedom to set the d_j coefficients equal to their classical values in (4.9). In the first case $d_0 = d_2 = 0$, d_1 arbitrary but not vanishing, the equations in (4.8) lead to

$$\begin{aligned} 1+\frac{5}{2}\alpha+\frac{55}{36}\alpha^2+\frac{5}{12}\alpha^3+\frac{\alpha^4}{18}+(1+\alpha+\frac{\alpha^2}{4})G+(3+3\alpha+\frac{\alpha^2}{4})H+ \\ -(2+\frac{1}{2}\alpha)I-\frac{1}{2}\alpha J=0 \end{aligned} \tag{4.12}$$

This condition does not determine the coefficients uniquely. It is satisfied for example by the non singular (in the limit $\alpha \to 0$) solution

$$H=I=J=0 \qquad\qquad G=-\frac{1+\frac{5}{2}\alpha+\frac{55}{36}\alpha^2+\frac{5}{12}\alpha^3+\frac{\alpha^4}{18}}{1+\alpha+\frac{\alpha^2}{4}} \tag{4.13}$$

We observe that even if the quantum corrections require the presence of a total derivative term nonvanishing in the classical limit, this, as already emphasized, does not alter the charge conserved at the classical level. These perturbative contributions modify the conservation at the quantum level and are actually necessary to implement an exact symmetry of the theory.

Exactly the same conclusions can be reached for the second choice in (4.9), $d_1 = \pm\sqrt{2}$, d_0 and d_2 arbitrary and non zero. Now the system in (4.8) becomes

$$\begin{aligned} &3H-G=0 \qquad\qquad (4.14)\\ &6\alpha+\frac{14}{3}\alpha^2-\frac{13}{9}\alpha^3-\alpha^4-\frac{2}{9}\alpha^5+4\alpha(1+\alpha+\frac{\alpha^2}{3})G+4\alpha(1+\alpha)I=0 \\ &\frac{\alpha}{4}d_0^2+\frac{\alpha}{2}\left[1+\frac{5}{2}\alpha+\frac{55}{36}\alpha^2+\frac{5}{12}\alpha^3+\frac{\alpha^4}{18}+(2+2\alpha+\frac{\alpha^2}{3})G-(2+\frac{1}{2}\alpha)I-\frac{1}{2}\alpha J\right]=0 \\ &\frac{\alpha}{4}d_2^2+\frac{\alpha}{2}\left[1+\frac{5}{2}\alpha+\frac{55}{36}\alpha^2+\frac{5}{12}\alpha^3+\frac{\alpha^4}{18}+(2+2\alpha+\frac{\alpha^2}{3})G-(2+\frac{1}{2}\alpha)I-\frac{1}{2}\alpha J\right]=0 \end{aligned}$$

Therefore, first we need impose a further restriction $d_0^2 = d_2^2$ with respect to the classical result, then we have to solve the remaining two equations in three unknowns. Again acceptable solutions exist, not uniquely determined.

4.2 The $c_2^{(1)}$ system

We choose the following representation for the simple roots, $\vec{\alpha}_1 = \sqrt{2}(0,1)$, $\vec{\alpha}_2 = \sqrt{2}(1,-1)$ so that in terms of two scalar fields the interactions become

$$V = e^{-\sqrt{2}(\phi_1+\phi_2)} + 2e^{\sqrt{2}\phi_2} + e^{\sqrt{2}(\phi_1-\phi_2)} \tag{4.15}$$

and

$$B = d_0\ e^{-\frac{1}{\sqrt{2}}(\phi_1+\phi_2)} + d_1\ e^{\frac{1}{\sqrt{2}}\phi_2} + d_2\ e^{\frac{1}{\sqrt{2}}(\phi_1-\phi_2)} \tag{4.16}$$

The action is symmetric under the exchange $\phi_1 \to -\phi_1$. Also for this nonsimply laced theory the first nontrivial high–spin conserved current in the bulk region is at spin 4. From the general expression in (2.37) and the bulk conservation equations (2.39), we find the coefficients of $J^{(4)}$ (see also Ref. [4])

$$\begin{aligned} a_{1111} &= a_{2222} = 4 \qquad\qquad a_{1122} = -4(1+\alpha) \qquad\qquad b_{112} = -6\sqrt{2}(2+3\alpha+\alpha^2) \\ d_{11} &= -(8+24\alpha+23\alpha^2+6\alpha^3) \qquad\qquad d_{22} = 4+6\alpha+\alpha^2 \end{aligned} \tag{4.17}$$

while $c_{112} \equiv G$, $c_{222} \equiv 3H$, $e_{11} \equiv I$ and $e_{22} \equiv J$ are still undetermined at this stage. The quantum trace can be computed explicitly using in (2.38) the values just obtained for the coefficients (4.17)

$$\Theta^{(4)} = \Theta_0 + \Theta_1 + \Theta_2 \tag{4.18}$$

with

$$\begin{aligned} \Theta_0 \ = \ & \Bigg\{ \left[-\left(2+\frac{17}{3}\alpha+\frac{9}{2}\alpha^2+\frac{5}{6}\alpha^3\right) + \frac{\alpha^2}{2\sqrt{2}}(G+H) - \frac{\alpha}{2}(I+J) \right] (\partial\phi_1+\partial\phi_2)^2 + \\ & + \left[\frac{1}{\sqrt{2}}(1+2\alpha)G - I \right] (\partial\phi_1)^2 + \\ & + \left[\frac{\alpha}{\sqrt{2}}G + \frac{3}{\sqrt{2}}(1+\alpha)H - J \right] (\partial\phi_2)^2 + \\ & + \left[\sqrt{2}\left(1+\frac{3}{2}\alpha\right)G + \frac{3}{\sqrt{2}}\alpha H - I - J \right] \partial\phi_1\partial\phi_2 \ + \\ & - \left[\frac{\alpha}{\sqrt{2}}\left(\frac{1}{3}-\frac{\alpha}{2}-\frac{5}{6}\alpha^2\right) + \frac{\alpha^2}{4}(G+H) + \frac{\alpha}{2\sqrt{2}}(I+J) \right] (\partial^2\phi_1+\partial^2\phi_2) + \\ & + \frac{1}{\sqrt{2}}I\ \partial^2\phi_1 + \frac{1}{\sqrt{2}}J\ \partial^2\phi_2 \Bigg\} e^{-\sqrt{2}(\phi_1+\phi_2)} \end{aligned}$$

$$\begin{aligned} \Theta_1 \ = \ & \Big\{ \left[6(2+3\alpha+\alpha^2) - \sqrt{2}G \right] (\partial\phi_1)^2 + \\ & + \left[-2\left(2+\frac{10}{3}\alpha+\frac{3}{2}\alpha^2+\frac{\alpha^3}{6}\right) - \sqrt{2}\left(3+3\alpha+\frac{\alpha^2}{2}\right)H - 2\left(1+\frac{\alpha}{2}\right)J \right] (\partial\phi_2)^2 + \end{aligned}$$

$$- \left[\frac{\sqrt{2}}{3}\left(\alpha + \frac{3}{2}\alpha^2 + \frac{\alpha^3}{2}\right) + \frac{\alpha^2}{2}H + \sqrt{2}\left(1+\frac{\alpha}{2}\right)J\right]\partial^2\phi_2\Big\} e^{\sqrt{2}\phi_2}$$

$$\Theta_2 \;=\; \Theta_0(\phi_1 \to -\phi_1) \tag{4.19}$$

In this case the equations at the boundary (2.41) are

$$\begin{aligned}
&\alpha(G-3H)d_0 = 0\\
&(6+\frac{3}{2}\alpha - \frac{9}{2}\alpha^2)d_1 d_0^2 - [12 + 23\alpha + \frac{41}{3}\alpha^2 + \frac{9}{2}\alpha^3 + \frac{5}{6}\alpha^4 +\\
&\qquad\qquad - \sqrt{2}\alpha(3+3\alpha+\alpha^2)H + \frac{\alpha}{2}(1+\alpha)I + \frac{\alpha^2}{2}J]d_1 = 0\\
&3\alpha^2 d_1^2 d_0 - \alpha[16 + \frac{80}{3}\alpha + 6\alpha^2 - \frac{2}{3}\alpha^3 - \sqrt{2}(6+6\alpha+\alpha^2)H - 2(2+\alpha)I]d_0 = 0\\
&\alpha(G-3H)d_2 = 0\\
&(6+\frac{3}{2}\alpha - \frac{9}{2}\alpha^2)d_1 d_2^2 - [12 + 23\alpha + \frac{41}{3}\alpha^2 + \frac{9}{2}\alpha^3 + \frac{5}{6}\alpha^4 +\\
&\qquad\qquad - \sqrt{2}\alpha(3+3\alpha+\alpha^2)H + \frac{\alpha}{2}(1+\alpha)I + \frac{\alpha^2}{2}J]d_1 = 0\\
&3\alpha^2 d_1^2 d_2 - \alpha[16 + \frac{80}{3}\alpha + 6\alpha^2 - \frac{2}{3}\alpha^3 - \sqrt{2}(6+6\alpha+\alpha^2)H - 2(2+\alpha)I]d_2 = 0
\end{aligned} \tag{4.20}$$

It is rather easy to see that the situation is very similar to the one described in detail for the previous example. First we verify that in the classical limit G, H, I and J do not play any role and we recover the classical solution for the boundary coefficients

$$\begin{aligned}
&a) \quad d_1 = 0 && d_0,\ d_2 \quad arbitrary\\
&b) \quad d_0 = d_2 = \pm\sqrt{2} && d_1 \quad arbitrary
\end{aligned} \tag{4.21}$$

Then we observe that at the quantum level no consistent solution can be found without introducing total derivative terms in the current. Indeed, setting the total derivatives to zero, the system in (4.20) becomes

$$\begin{aligned}
&(6+\frac{3}{2}\alpha - \frac{9}{2}\alpha^2)d_1 d_0^2 - [12 + 23\alpha + \frac{41}{3}\alpha^2 + \frac{9}{2}\alpha^3 + \frac{5}{6}\alpha^4]d_1 = 0\\
&3\alpha^2 d_1^2 d_0 - \alpha[16 + \frac{80}{3}\alpha + 6\alpha^2 - \frac{2}{3}\alpha^3]d_0 = 0\\
&(6+\frac{3}{2}\alpha - \frac{9}{2}\alpha^2)d_1 d_2^2 - [12 + 23\alpha + \frac{41}{3}\alpha^2 + \frac{9}{2}\alpha^3 + \frac{5}{6}\alpha^4]d_1 = 0\\
&3\alpha^2 d_1^2 d_2 - \alpha[16 + \frac{80}{3}\alpha + 6\alpha^2 - \frac{2}{3}\alpha^3]d_2 = 0
\end{aligned} \tag{4.22}$$

The pattern is the same as in the $d_3^{(2)}$ case: one solution is given by $d_0 = d_1 = d_2 = 0$, that is a vanishing potential at the boundary. The other solution corresponds to the

following values for the d_j coefficients

$$
\begin{aligned}
d_1^2 &= \frac{1}{3\alpha}(16+\frac{80}{3}\alpha+6\alpha^2-\frac{2}{3}\alpha^3) \\
d_0^2 = d_2^2 &= 2+\frac{20\alpha+\frac{68}{3}\alpha^2+\frac{9}{2}\alpha^3+\frac{5}{6}\alpha^4}{6+\frac{3}{2}\alpha-\frac{9}{2}\alpha^2}
\end{aligned}
\tag{4.23}
$$

Again the results in (4.23) are not interesting, in particular d_1 is singular in the classical limit. The way to circumvent the problem is to include the total derivative terms and reexamine the original set of equations in (4.20). It is clear that one possibility is to solve the system with the d_j boundary coefficients fixed at their classical values (4.21). For $d_1 = 0$, d_0 and d_2 arbitrary and non zero, we find

$$
\begin{aligned}
& G-3H=0 \\
& 16+\frac{80}{3}\alpha+6\alpha^2-\frac{2}{3}\alpha^3-\sqrt{2}(6+6\alpha+\alpha^2)H-2(2+\alpha)I=0
\end{aligned}
\tag{4.24}
$$

For $d_0 = d_2 = \pm\sqrt{2}$, d_1 arbitrary and nonzero, (4.20) gives

$$
\begin{aligned}
& (G-3H)=0 \\
& 20\alpha+\frac{68}{3}\alpha^2+\frac{9}{2}\alpha^3+\frac{5}{6}\alpha^4+\sqrt{2}\alpha(3+3\alpha+\alpha^2)H-\frac{\alpha}{2}(1+\alpha)I-\frac{\alpha^2}{2}J=0 \\
& 3\alpha^2 d_1^2-\alpha[16+\frac{80}{3}\alpha+6\alpha^2-\frac{2}{3}\alpha^3-\sqrt{2}(6+6\alpha+\alpha^2)H-2(2+\alpha)I]=0
\end{aligned}
\tag{4.25}
$$

It is worth emphasizing that in both cases not all the constants $G,\ldots,J$ can be set to zero not even in the classical limit $\alpha\to 0$. The quantum corrections feed back into the classical results, requiring the presence of nonvanishing total derivative terms in the spin–4 current.

5 Conclusions

We have studied the quantum properties of higher–spin charges for affine Toda theories defined on the upper plane in the presence of a nontrivial perturbation at the border. We have attempted a systematic analysis of the various theories, but we had to face the complexity of the algebraic manipulations which an exact quantum calculation requires. Moreover the diverse behaviour of different systems and the diverse behaviour of different spin currents within the same system have prevented us from completeness. With these caveats, we have accomplished nonetheless several goals and obtained quite interesting and unexpected results.

First we have developed a general technique which allows to address the problem of quantum charge conservation for a system defined on a manifold with boundary. This method, even if perturbative in spirit, allows to obtain exact answers, to all loop orders in perturbation theory. The algebraic difficulty of its application arises when dealing with higher and higher–spin currents: however since the procedure is a step by step one it can be implemented with a computer program.

Second we have tested our approach on several examples, general enough to illustrate the issues we wished to discuss. Although we have not found results with a repetitive pattern, the procedure itself is repetitive and in a sense straightforward to be applied.

Finally we have shown that in the presence of a nonvanishing perturbation at the boundary the construction of quantum conserved charges is not automatically guaranteed by the existence of a corresponding classical charge. At the quantum level total derivative terms added to the currents become relevant and necessary, in certain cases, for the realization of global, exact symmetries. This feature is a peculiar property of systems defined on manifolds with a non trivial boundary potential.

Perhaps the most striking finding of our study has been the realization that for the $a_n^{(1)}$ affine Toda theories there is no choice of a nonvanishing boundary perturbation, no possibility of a quantum redefinition of the current which allow the quantum existence of certain higher–spin conserved charges. The first classical conservation spoiled at the quantum level by anomalies that cannot be cured is at spin 4. We have checked this failure of the conservation law on explicit examples and we expect the same happening at higher spin too.

We have not attempted to repeat this last analysis on nonsimply laced Toda systems. It would seem that for the $d_3^{(2)}$ and $c_2^{(1)}$ theories similar anomalies might appear at spin > 4.

In any case, for all the theories we have considered, at least one higher–spin conserved charge has been found and this is sufficient to imply the existence of factorizable, elastic S–matrices [16]. It would be interesting to proceed in this direction and make precise the correspondence between S matrices and boundary Toda systems. In particular one would like to understand the role played by the *quantum* form of the boundary potential of the $a_n^{(1)}$ theories in the specific construction of an exact S–matrix.

A Appendix

In this Appendix we show with an explicit example how to proceed in the computations of local contributions to $\langle J^{(1)}\rangle|_{x_1=0}$. Let us consider

$$i\left\langle\left(\partial^3\phi_a\partial\phi_b-\bar\partial^3\phi_a\bar\partial\phi_b\right)e^{-\frac{1}{2\pi\alpha}\int d^2w\ V+\frac{1}{2\pi\alpha}\int dw_0\ B}\right\rangle_0 \tag{A.1}$$

We want to extract the local terms when this expression is evaluated at the border $x_1=0$. First, using the definitions in eq. (2.2) we write

$$\begin{aligned} i\left(\partial^3\phi_a\partial\phi_b-\bar\partial^3\phi_a\bar\partial\phi_b\right) &= \frac{1}{2}\left(-\partial_1^3\phi_a\partial_0\phi_b-3\partial_0\partial_1^2\phi_a\partial_1\phi_b\right.\\ &\qquad\left.+\partial_0^3\phi_a\partial_1\phi_b+3\partial_0^2\partial_1\phi_a\partial_0\phi_b\right)\\ &= 2\partial_0^2\partial_1\phi_a\partial_0\phi_b+2\partial_0^3\phi_a\partial_1\phi_b\\ &\qquad-3\partial_0\partial\bar\partial\phi_a\partial_1\phi_b-\partial_1\partial\bar\partial\phi_a\partial_0\phi_b \end{aligned} \tag{A.2}$$

In the last equality we have written $\partial_1^2=2\partial\bar\partial-\partial_0^2$ so that it is easier to identify in (A.1) the Wick contractions which lead to local contributions. Indeed we use

$$\begin{aligned} \lim_{x_1\to 0}\langle\partial_1\phi(x_0,x_1)\phi(w_0,0)\rangle_0 &= \lim_{x_1\to 0}\ -\alpha\left[\frac{i}{x_0-w_0+ix_1}-\frac{i}{x_0-w_0-ix_1}\right]\\ &= -2\pi\alpha\delta^{(1)}(x_0-w_0) \end{aligned} \tag{A.3}$$

so that we immediately obtain

$$\left\langle 2\partial_0^2\partial_1\phi_a\partial_0\phi_b+2\partial_0^3\phi_a\partial_1\phi_b\right\rangle\Big|_{x_1=0}\sim -2\partial_0^2B_a\partial_0\phi_b-2B_b\partial_0^3\phi_a \tag{A.4}$$

For the other two terms in (A.2) we need consider a double expansion in the bulk and boundary potentials. We have

$$\begin{aligned} &\left\langle\partial_0\partial\bar\partial\phi_a\partial_1\phi_b e^{-\frac{1}{2\pi\alpha}\int d^2w\ V+\frac{1}{2\pi\alpha}\int dw_0'\ B}\right\rangle_0\sim\\ &\sim-\frac{1}{(2\pi\alpha)^2}\int d^2w\frac{\partial V}{\partial\phi_a}\left(-\frac{\alpha}{2}\right)\partial_{x_0}\partial\bar\partial\left(\log 2|x-w|^2+\log 2|x-\bar w|^2\right)\\ &\qquad\int dw_0'\frac{\partial B}{\partial\phi_b}(-\alpha)\left[\frac{i}{x_0-w_0'+ix_1}-\frac{i}{x_0-w_0'-ix_1}\right] \end{aligned} \tag{A.5}$$

Using in the upper–half plane

$$\partial\bar\partial\left(\log 2|x-w|^2+\log 2|x-\bar w|^2\right)=2\pi\delta^{(2)}(x-w) \tag{A.6}$$

and eq.(A.3) we obtain the result

$$\left\langle \partial_0 \partial \bar{\partial} \phi_a \partial_1 \phi_b \right\rangle \sim -\frac{1}{2} \partial_0 V_a B_b \tag{A.7}$$

The last term is treated in similar manner

$$\begin{aligned}
&\left\langle \partial_1 \partial \bar{\partial} \phi_a \partial_0 \phi_b e^{-\frac{1}{2\pi\alpha}\int d^2w \; V + \frac{1}{2\pi\alpha}\int dw_0' \; B} \right\rangle_0 \sim \\
&\quad \sim -\frac{1}{(2\pi\alpha)^2} \partial_0 \phi_b \int d^2w \frac{\partial^2 V}{\partial \phi_a \partial \phi_c} \int dw_0' \frac{\partial B}{\partial \phi_c} \left(-\frac{\alpha}{2}\right) \log 2|w - w_0'|^2 \\
&\quad \left(-\frac{\alpha}{2}\right) \partial \bar{\partial} \left[\frac{i}{x_0 - w_0 + i(x_1 - w_1)} + \frac{i}{x_0 - w_0 + i(x_1 + w_1)} + \right. \\
&\quad \left. - \frac{i}{x_0 - w_0 - i(x_1 - w_1)} - \frac{i}{x_0 - w_0 - i(x_1 + w_1)} \right]
\end{aligned} \tag{A.8}$$

Integrating by parts the $\partial \bar{\partial}$ derivatives we obtain a local contribution when they hit the log and produce a $\delta^{(2)}(w - w_0')$, while the terms in the square bracket, when evaluated in the limit $x_1 \to 0$, give rise to a $\delta^{(1)}(x_0 - w_0')$. We then obtain

$$\left\langle \partial_1 \partial \bar{\partial} \phi_a \partial_0 \phi_b \right\rangle \sim -\frac{1}{2} V_{ac} B_c \partial_0 \phi_b \tag{A.9}$$

Summing all the contributions the total result quoted in (2.27) is recovered.

This work has been partially supported by grants no. SC1–CT92–0789 and no. CEE–CHRX–CT92–0035.

References

[1] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. **120** (1979) 253;
A.E. Arinshtein, V.A. Fateev and A.B. Zamolodchikov, Phys. Lett. **B87** (1979) 389;
J.L. Cardy and G. Mussardo, Phys. Lett. **B225** (1989) 275;
P.G.O. Freund, T.R. Klassen and E. Melzer, Phys. Lett. **B229** (1989) 243;
C. Destri and H.J. de Vega, Phys. Lett. **B233** (1989) 336;
G. Mussardo and G. Sotkov, in "Recent developments in Conformal Field Theories", Trieste, Italy, October 2–4, 1989;
H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, Phys. Lett. **B227** (1989) 411;
P. Christe and G. Mussardo, Nucl. Phys. **B330** (1990) 465;
T.R. Klassen and E. Meltzer, Nucl. Phys. **B338** (1990) 485;
G.W. Delius, M.T. Grisaru and D. Zanon, Phys. Lett. **B277** (1992) 414; Nucl. Phys. **B382** (1992) 365.

[2] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, Nucl. Phys. **B338** (1990) 689.

[3] P.P. Kulish and E.R. Nisimov, Theor. Math. Phys. **29** (1976) 922;
R. Flume, Phys. Lett. **62B** (1976) 93; Phys. Lett. **68B** (1977) 487;
J.H. Lowenstein and E.R. Speer, Comm. Math. Phys. **63** (1978) 97;
T. Marinucci and S. Sciuto, Nucl. Phys. **B156** (1979) 144;
E.R. Nisimov, Nucl. Phys. **B163** (1980) 374;
R. Sasaki and I. Yamanaka, Adv. Stud. Pure Math. **16** (1988) 271;
I. Yamanaka and R. Sasaki, Prog. Theor. Phys. **79** (1988) 1167.

[4] G.W. Delius, M.T. Grisaru and D. Zanon, Nucl. Phys. **B385** (1992) 307.

[5] S. Penati and D. Zanon, Phys. Lett. **288B** (1992) 297;
S. Penati, A. Gualzetti and D. Zanon, Nucl. Phys. **B398** (1993) 622;
S. Penati and D. Zanon, in *"String theory, quantum gravity and the unification of the fundamental interactions"* (Rome, September 1992), eds. M. Bianchi et al., World Scientific (1993) 450, hep–th/9211033.

[6] J.L. Cardy, Nucl. Phys. **B240[FS12]** (1984) 514; Nucl. Phys. **B324** (1989) 581.

[7] S. Goshal and A.B. Zamolodchikov, Int. J. Mod. Phys. **A9** (1994) 3841.

[8] A.O. Caldeira and A.J. Leggett, Physica **121A** (1983) 587; Phys. Rev. Lett.**46** (1981) 211; Ann. of Phys. **149** (1983) 374;

C.G. Callan and L. Thorlacius, Nucl. Phys. **B329** (1990) 117;
I. Affleck, Nucl. Phys. **B336** (1990) 517;
I. Affleck and A.W.W. Ludwig, Nucl. Phys. **B352** (1991) 849; **B360** (1991) 641; Phys. Rev. Lett. **67** (1991) 161, 3160; **68** (1992) 1046;
P. Fendley, A. Ludwig and H. Saleur, cond–mat/9408068;
A. Abouelsaood, C.G. Callan, C.R. Nappi and S.A. Yost, Nucl. Phys. **B280[FS18]** (1987) 599.

[9] E. Corrigan, P. E. Dorey, R.H. Rietdijk and R. Sasaki, Phys. Lett. **B333** (1994) 83;
E. Corrigan, P. E. Dorey and R.H. Rietdijk, to appear in Proceedings of *"Quantum Field Theory, Integrable Models and beyond"*, Kyoto, Japan, Feb. 14–17, 1994, hep–th/9407148.

[10] P. Bowcock, E. Corrigan, P.E. Dorey and R.H. Rietdijk, Nucl. Phys. **B445** (1995) 469.

[11] S. Penati and D. Zanon, Phys. Lett. **B358** (1995) 63.

[12] P. Fendley and H. Saleur, Nucl. Phys. **B430** (1994) 577;
M. Ameduri, R. Konik and A. LeClair, Phys. Lett. **B354** (1995) 376;
Z–M Sheng and H–B Gao, "On the sine–Gordon – Thirring equivalence in the presence of a boundary", DTP–95/43, hep–th/9512011;
L. Mezincescu, private communication.

[13] A. Fring and R. Koberle, Nucl. Phys. **B421** (1994) 159; Nucl. Phys. **B419** (1994) 647;
R. Sasaki, in *"Interface between Physics and Mathematics"*, eds. W. Nahm and J–M. Shen, World Scientific (1994) 201, hep–th/9311027;
J.D. Kim, "Boundary Reflection Matrix for *ade* Affine Toda Field Theory", DTP/95–31, hep–th/9506031.

[14] S. Penati, A. Refolli and D. Zanon, "Quantum boundary currents for nonsimply laced Toda theories" , IFUM–518–FT, hep–th/9510084, to appear in Phys. Lett. **B**.

[15] A.V. Mikhailov, M.A. Olshanetsky and A.M. Perelemov, Comm. Math. Phys. **79** (1981) 473;
D. Olive and N. Turok, Nucl. Phys. **B257**[FS14] (1985) 277.

[16] A. Polyakov, Phys. Lett. **72B** (1977) 224 ;
S. Parke, Nucl. Phys. **B174** (1980) 166.